3. Quantisation as an eigenvalue problem; by E. Schrödinger*

(first communication.)

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§ 1. In this communication I would like first to show, in the simplest case of the (non-relativistic and unperturbed) hydrogen atom, that the usual prescription for quantisation can be substituted by another requirement in which no word about "integer numbers" occurs anymore. Rather, the integerness[1] emerges in the same natural way as, for example, the integerness of the *number of knots* of a vibrating string. The new interpretation is generalisable and touches, as I believe, very deeply the true essence of the quantisation prescription.

The usual form of the latter is tied to the Hamiltonian partial differential equation:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \ . \tag{1}$$

It is looked for a solution of this equation that appears as a *sum* of functions, each of only one of the independent variables q.

We introduce now in place of S a new, unknown function ψ in such a manner that ψ would appear as a *product* of suitable functions of the single coordinates. That is, we set:

$$S = K \lg \psi \ . \tag{2}$$

The constant K must be introduced for dimensional reasons and it has the dimension of an *action*. With this one obtains:

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E \ . \tag{1'}$$

We *do not* look now for a solution of equation (1′), but we stipulate the following requirement. Neglecting the variability of the masses, or considering it at least as long as the *single* electron problem is concerned, equation (1′) can always be brought to the form: a quadratic form for ψ and its first derivatives $= 0$. We look for such

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[1]**Translator's note:** I have translated here *Ganzzahligkeit* as "integerness", meaning "the property of a number being integer".

real, single-valued in the whole configuration space, finite and twice continuously differentiable functions ψ, which make to an *extremum* the integral, extended over the whole configuration space, of the just mentioned quadratic form[2]). *With this variation problem we substitute the quantum conditions.*

First of all, we will take for H the Hamiltonian function of the Keplerian motion and show that the established requirement is satisfiable for *all positive*, but only for a *discrete set of negative*, values of E. That is, the mentioned variation problem has a discrete and a continuous spectrum of eigenvalues. The discrete spectrum corresponds to the Balmer terms, whereas the continuous one corresponds to the energies of the hyperbolic orbits. In order for numerical agreement to exist, K must get the value $h/2\pi$.

Since for the arrangement of the variation equations the choice of coordinates is of no importance, we choose the right-angled Cartesian ones. Then, in our case, (1′) reads (e, m are the electron charge and mass):

$$(1'') \qquad \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0 \,.$$

$$r = \sqrt{x^2 + y^2 + z^2} \,.$$

And our variation problem reads:

$$(3) \qquad \delta J = \delta \int\int\int dxdydz \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2\right] = 0 \,,$$

with the integral extended over the whole space. One finds from this, in the usual way:

$$(4) \qquad \frac{1}{2}\delta J = \int df\delta\psi\frac{\partial\psi}{\partial n} - \int\int\int dxdydz\delta\psi\left[\Delta\psi + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi\right] = 0 \,.$$

It must then be, firstly, that

$$(5) \qquad \Delta\psi + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0 \,,$$

and, secondly, the integral to be extended over the infinitely far closed surface must be

$$(6) \qquad \int df\delta\psi\frac{\partial\psi}{\partial n} = 0 \,.$$

(It will turn out that because of the latter requirement we have to supplement our variation problem with still a requirement over the behaviour of $\delta\psi$ at infinity, so that also the above claimed *continuous* spectrum actually exists. But, later about that.)

[2]It doesn't escape to me that this formulation is not completely unambiguous.

The solution of (5) can be worked out (*for example*) in spatial polar coordinates r, ϑ, φ, by setting ψ as a *product* of a function of r, a function of ϑ and a function of φ. The method is abundantly known. For the dependence on the polar angles it comes out a *spherical harmonic*,[3] whereas for the dependence on r — we want to call χ its function — one easily obtains the differential equation:

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left(\frac{2mE}{K^2} + \frac{2me^2}{K^2 r} - \frac{n(n+1)}{r^2}\right)\chi = 0 \,. \tag{7}$$

$$n = 0, 1, 2, \ldots \,.$$

The restriction of n to integer numbers is notoriously *necessary*, so that the dependence on the polar angles becomes *single-valued*. — We need solutions of (7) that remain finite for all real, non-negative values of r. Now, equation (7) has[4] in the complex r-plane *two* singularities, at $r = 0$ and at $r = \infty$, of which the second is an "irregular singular point"[5] (an essential singularity)[6] of *all* integrals,[7] whereas the first is not (for no integral). Both these singularities are forming the *boundary points of our real interval*. Now, in such case one knows that the requirement of *finiteness* for the function χ at the boundary points amounts to a *boundary condition*. The equation has *in general* no integral at all which at *both* boundary points stays finite, but such an integral exists only for certain specific values of the constants which appear in the equation. It is necessary to determine these specific values.

The fact just pointed out is the *salient* point in the whole investigation.

We consider first the singular point $r = 0$. The so-called *characteristic equation*,[8] which determines the behaviour of the integral at this point, is:[9]

$$\varrho(\varrho - 1) + 2\varrho - n(n+1) = 0 \tag{8}$$

with the roots:

$$\varrho_1 = n \,, \qquad \varrho_2 = -(n+1) \,. \tag{8'}$$

[3] **Translator's note:** in German, *Kugelflächenfunktion*. Literally, "spherical surface function".

[4] For the guide to the treatment of equation (7) I am obliged to the greatest gratitude to Hermann Weyl. For the claims which will not be proved in the following, I refer to L. Schlesinger, Differential equations (Collection Schubert Nr. 13, Göschen, 1900, especially chapters 3 and 5).

[5] **Translator's note:** in German, *Stelle der Unbestimmtheit*. Literally "location of the indefiniteness".

[6] **Translator's note:** in German, *wesentlich singuläre Stelle*. Literally, "essential singular location"

[7] **Translator's note:** "integral", here and in the following, stands for "solution of the differential equation" (which, in fact, is an integral).

[8] **Translator's note:** also known as "indicial equation". In German, *determinierende Fundamentalgleichung*, literally "fundamental determining equation".

[9] **Translator's note:** in order to find equation (8) one postulates a power series:

$$\chi(r) = r^{\varrho} \sum_{k=0}^{\infty} c_k r^k \,,$$

with $c_0 \neq 0$, and substitutes it into equation (7). The coefficient of the lowest power of r is the characteristic equation. This is generally known as the Frobenius method.

The two canonical integrals at this point go then with exponents n and $-(n+1)$. Since n is non-negative, only the former integral is useful for us. Since this goes with the *larger* exponent, it is represented by a usual power series that starts with r^n. (The other integral, which does not interest us, can contain a logarithm, due to the integer difference between the exponents.) Since the next singular point lies only at infinity, the mentioned power series converges continuously[10] and represents an *entire transcendental* function.[11] We thus establish:

The solution we looked for is (a part from an irrelevant constant factor) a single-valued, specific transcendental function, which at $r = 0$ goes with exponent n.

Now, the question is to investigate the behaviour of this function at the *infinity* of the positive real axis. To this purpose, we simplify equation (7) by means of the substitution

$$\chi = r^\alpha U \ , \tag{9}$$

in which α is chosen so that the term with $1/r^2$ cancels out. To this purpose, α must get one of the two values n, $-(n+1)$, as one easily verifies. Equation (7) takes then the form

$$\frac{d^2U}{dr^2} + \frac{2(\alpha+1)}{r}\frac{dU}{dr} + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)U = 0 \ . \tag{7'}$$

Its integrals go at $r = 0$ with exponents 0 and $-2\alpha - 1$.[12] For the first value of α, $\alpha = n$, is the *first*, for the second value of α, $\alpha = -(n+1)$, is the *second* of these integrals an entire transcendental function and leads via (9) to the *sought after* solution, which is certainly single-valued. Therefore, we lose nothing if we limit ourselves to *one* of the two values of α. We choose

$$\alpha = n \ . \tag{10}$$

Our solution U goes then at $r = 0$ with exponent 0. The mathematicians call equation (7′) a Laplace equation. The general type is

$$\frac{d^2U}{dr^2} + \left(\delta_0 + \frac{\delta_1}{r}\right)\frac{dU}{dr} + \left(\varepsilon_0 + \frac{\varepsilon_1}{r}\right)U = 0 \ . \tag{7''}$$

In our case, the constants have the values

$$\delta_0 = 0 \ , \quad \delta_1 = 2(\alpha+1) \ , \quad \varepsilon_0 = \frac{2mE}{K^2} \ , \quad \varepsilon_1 = \frac{2me^2}{K^2} \ . \tag{11}$$

This type of equation is relatively easy to handle for *the* reason that the so-called Laplace transform, which in general results *again* in an equation of *second* order, leads *here* to the first order, which is solvable through quadrature. This allows

[10] **Translator's note:** in German, *beständig konvergieren.*

[11] **Translator's note:** the function is "entire" because it has no singularities, except that at infinity, and it is "transcendental" because it is not a polynomial.

[12] **Translator's note:** In order to see this, one calculates again the characteristic equation.

for a representation of the solution of (7″) even through integrals in the complex plane. I quote here the final result only.[13]) The integral[14]

$$U = \int_L e^{zr}(z-c_1)^{\alpha_1-1}(z-c_2)^{\alpha_2-1}dz \tag{12}$$

is a solution of (7″) for an integration path L for which

$$\int_L \frac{d}{dz}\left[e^{zr}(z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\right]dz = 0\ . \tag{13}$$

The constants c_1, c_2, α_1, α_2 have the following values. c_1 and c_2 are the roots of the quadratic equation:

$$z^2 + \delta_0 z + \varepsilon_0 = 0 \tag{14}$$

and

$$\alpha_1 = \frac{\varepsilon_1 + \delta_1 c_1}{c_1 - c_2}\ , \qquad \alpha_2 = \frac{\varepsilon_1 + \delta_1 c_2}{c_2 - c_1}\ . \tag{14′}$$

In the case of equation (7′), one has then, according to (11) and (10)

$$\begin{cases} c_1 = +\sqrt{\dfrac{-2mE}{K^2}}\ , \qquad c_2 = -\sqrt{\dfrac{-2mE}{K^2}}\ ; \\ \alpha_1 = \dfrac{me^2}{K\sqrt{-2mE}} + n + 1\ , \qquad \alpha_2 = -\dfrac{me^2}{K\sqrt{-2mE}} + n + 1\ . \end{cases} \tag{14″}$$

The integral representation (12) allows not only to oversee the asymptotic behaviour of the general solution[15] when r goes to infinity in a certain way, but also to give this behaviour for a *specific* solution, which is always much harder.

We want now, first of all, to *exclude* the case in which α_1 and α_2 are real, integer numbers. When this happens, it happens always at the same time for both the two quantities and indeed in the case, and only in the case, when

$$\frac{me^2}{K\sqrt{-2mE}} = \text{real integer number}\ . \tag{15}$$

Therefore, we suppose that (15) is not satisfied.

The behaviour of the general solution for r going to infinity in a certain manner - we want always think of r going to the real, positive infinite - is thereupon[16]) characterised by the behaviour of the two linearly independent solutions which are obtained via the following *two specifications* of the integration path L and that we

[13]cf. L. Schlesinger, loc. cit. One owes the theory to H. Poincaré and J. Horn.

[14]**Translator's note:** I put in the appendix the derivation of equation (12).

[15]**Translator's note:** I have employed the terminology "general solution" for the translation of the German *Gesamtheit von Lösungen*, which literally translates as "totality of the solutions".

[16]When (15) is satisfied, at least one of the two integration paths described in the text becomes unusable, since it delivers a vanishing result.

want to dub U_1 and U_2. Let, for *both* instances, z come from infinity and go back there on the same path, and indeed in a direction such that

$$\lim_{z=\infty} e^{zr} = 0 \,, \tag{16}$$

i.e. the real part of zr should become negative infinite. Hereby condition (13) is satisfied. *In between*, let in *one* case (solution U_1) the point c_1, whereas in the *other* case (solution U_2) the point c_2, be circulated, each once.

These two solutions are now *asymptotically* (in the sense of Poincaré) represented, for very large real positive values of r, by

$$\begin{cases} U_1 \sim e^{c_1 r} r^{-\alpha_1} (-1)^{\alpha_1} \, (e^{2\pi i \alpha_1} - 1) \, \Gamma(\alpha_1) (c_1 - c_2)^{\alpha_2 - 1} \,, \\ U_2 \sim e^{c_2 r} r^{-\alpha_2} (-1)^{\alpha_2} \, (e^{2\pi i \alpha_2} - 1) \, \Gamma(\alpha_2) (c_2 - c_1)^{\alpha_1 - 1} \,, \end{cases} \tag{17}$$

where we content ourselves here with the first term of the asymptotic series, which goes on with negative integer powers of r.

We have now to distinguish the two cases $E \gtrless 0$.

Let it first be

1. $\underline{E > 0}$. We note first that hereby the non-satisfaction of (15) is guaranteed by the very fact that this quantity is purely imaginary. Further, according to (14″) also c_1 and c_2 becomes purely imaginary. The exponential functions in (17) are then, since r is real, finite periodic functions. The values of α_1 and α_2 show, according to (14″), that U_1 and U_2 *both* go to zero as r^{-n-1}. *The same must hold true also for our entire transcendent solution* U, whose behaviour we look for, *as it can always be built as a linear combination of* U_1 *and* U_2. Further, (9) shows, considering (10), that the function χ, i.e. the entire transcendent solution of the *originally existing* equation (7), still goes to zero as $1/r$, since it emerges from U via multiplication with r^n. We can therefore claim:

The Eulerian differential equation (5) *of our variational problem has for each positive* E *solutions which in the whole space are single-valued, finite and continuous and go to zero at infinity as* $1/r$ *with steady oscillations.* - It will have to be discussed yet about the surface condition (6).

2. $\underline{E < 0}$. In this case the condition (15) is not *eo ipso* excluded, but for the moment we hold onto its exclusion, as arranged. Then, according to (14″) and (17), U_1 grows unlimited for $r = \infty$, whereas U_2 vanishes *exponentially.* Our entire transcendent U (and the same holds true for χ) will then remain finite if and only if U is identical to U_2, a part from a numerical factor. *However, this is not the case.* One realises this so: choosing in (12) for the integration contour L a *closed* circuit about *both* the points c_1 and c_2 (due to the integerness of the *sum* $\alpha_1 + \alpha_2$ such contour is then *really closed* on the Riemann surface of the integrand), upon fulfillment of the very condition (13), one can easily show that the integral (12) represents then *our entire transcendent* U. Namely, it can be developed in a series of positive powers of r, which always converges for sufficiently small values of r, hence it satisfies the differential equation (7′), and thus the power series must coincide with that for U. So: U is represented by (12) when L is a closed contour about both the points c_1 and c_2. However, this closed contour can be distorted so that it appears as the *combination* of the two previously considered integration

paths, those related to U_1 and U_2, and indeed *without vanishing factors*, say 1 and $e^{2\pi i\alpha_1}$. QED[17]

Then, our entire transcendent U, which is the only possible solution of the variation problem among the solutions of (7′), does *not* stay finite for large r, under the prescriptions made. — Under reserve of the investigation of the *completeness*, that is of the proof that our procedure allows to find *all* the linearly independent solutions, we can then claim:

For those negative values of E which do not satisfy condition (15)*, our variation problem has no solution.*

We have now to investigate only that discrete set of negative values of E which *do satisfy condition* (15). α_1 and α_2 are then both integer. Of the two integration paths which have earlier delivered to us the fundamental system U_1, U_2, the first must certainly be modified in order to yield something non-vanishing. Since $\alpha_1 - 1$ is certainly positive, the point c_1 is then now neither a branch point nor a pole of the integrand, but an ordinary zero. The point c_2 can also become regular, if indeed also $\alpha_2 - 1$ is non-negative. In *any* case, however, two suitable integration paths can be easily provided and the integration along them can be even carried out in a closed form, through known functions, so that the behaviour of the solutions is completely controlled.

Let namely

$$(15') \qquad \frac{me^2}{K\sqrt{-2mE}} = l\ ; \qquad l = 1, 2, 3, 4 \ldots\ .$$

Then, according to (14″)

$$(14''') \qquad \alpha_1 - 1 = l + n\ , \qquad \alpha_2 - 1 = -l + n\ .$$

One has now to distinguish the two cases $l \leq n$ and $l > n$. To begin with, let

a) $l \leq n$. Then c_2 *and* c_1 lose any singular character, but gain the suitability to serve as initial or endpoints of the integration path, in order to satisfy condition (13). A third point suitable for this is the the negative real infinite. Any path between two of these three points delivers a solution and of these three solutions two by two are linearly independent, as one easily verifies by calculating the integral in closed form. In particular, the *entire transcendent solution* is delivered through the integration path from c_1 to c_2. Then, one recognises, without calculating it, that *this* integral stays regular for $r = 0$. I stress this because the actual calculation is rather likely to conceal this fact. On the other hand, *it* shows that the integral grows beyond any limit for positive infinitely large r. For large r, one of the *other* two integrals remains *finite*, but in turn it becomes infinite for $r = 0$.

So, we get in the case $l \leq n$ *no* solution to the problem.

b) $l > n$. Then, according to (14″), c_1 is a zero and c_2 is a pole of at least first order of the integrand. Two independent integrals are then delivered: one via the path that from $z = -\infty$ leads to the zero, taking care of avoiding the pole; the other by the *residual* of the pole. *The latter* is the entire transcendent. We want to give its calculated value, but at once multiplied by r^n, so that we obtain,

[17] **Translator's note:** in German, *w.z.b.w.*: *Was zu beweisen war*, i.e. "which was to demonstrate".

according to (9) and (10) the solution of the originally presented equation (7). (The irrelevant multiplicative constant is arbitrarily adjusted.) One finds:

$$\chi = f\left(r\frac{\sqrt{-2mE}}{K}\right)\ , \qquad f(x) = x^n e^{-x}\sum_{k=0}^{l-n-1}\frac{(-2x)^k}{k!}\binom{l+n}{l-n-1-k}\ . \tag{18}$$

One recognises that this is really a usable solution, since it remains finite for all real, non-negative r. Furthermore, the surface condition (6) is guaranteed by its exponential vanishing. We summarise the results for negative E:

For negative E, our variation problem has solutions in the case, and only in the case, in which E satisfies condition (15)*. To the integer number n, which gives the order of the spherical harmonic appearing in the solution, can be then always assigned only values smaller than l (of which there is at least one always available). The part of the solution dependent on r is given by* (18).[18]

By counting out the constants in the spherical harmonics ($2n+1$, as is well known) one finds further:

The solution found contains, for an admissible combination (n, l), exactly $2n+1$ arbitrary constants; for a given value of l then l^2 constants.[19]

We have thus proved in the main traits the claims established at the beginning, about the spectrum of the eigenvalues of our variation problem, although there are still gaps.

In the first place, the completeness of the *whole* established system of eigenfunctions. I do not want to deal with that in this note. According to other experiences one can suspect that no eigenvalues have escaped to us.

In the second place, it is to be remembered that the eigenfunctions established for positive E do not readily solve the variation problem in the form in which it was posed at the beginning, because at infinity they go to zero only as $1/r$, thus $\partial\psi/\partial r$ on a large sphere goes only as $1/r^2$ to zero. The surface integral (6) remains then just of the order of $\delta\psi$ at infinity. If one really wishes to obtain also the continuous spectrum, one must then add one extra condition to the *problem*: for example that $\delta\psi$ vanishes at infinity, or at least that it should tend to a constant value independent from the direction along which one goes to spatial infinity; in the latter case the spherical harmonics make the surface integral to vanish.

§ 2. Condition (15) gives

$$-E_l = \frac{me^4}{2K^2l^2}\ . \tag{19}$$

Therefore, the well-known Bohr energy levels, which correspond to the Balmer terms, arise when one assign to the constant K, which we had to introduce in (2) for dimensional reasons, the value

$$K = \frac{h}{2\pi}\ . \tag{20}$$

[18] **Translator's note:** Note that in the nowadays textbooks one usually sees $n \leftrightarrow l$, i.e. l is the order of the spherical harmonic and n denotes the energy levels.

[19] **Translator's note:** Because $\sum_{n=0}^{l-1}(2n+1) = l^2$.

Then one has of course

$$-E_l = \frac{2\pi^2 m e^4}{h^2 l^2} \ . \tag{19'}$$

Our l is the principal quantum number. $n+1$ has analogy with the azimuthal quantum number, the further division of this number in the definition of the spherical harmonics can be put into analogy with the division of the azimuthal quantum in a "equatorial" quantum and in a "polar" quantum. These numbers determine *here* the system of nodal lines on the sphere. Also the "radial quantum number," $l-n-1$ determines exactly the number of "nodal spheres", since one can easily convince themselves that the function $f(x)$ in (18) has exactly $l-n-1$ real positive roots. — The positive values of E correspond to the continuum of the hyperbolic trajectories, to which one can attribute, in a certain sense, the radial quantum number ∞. To this corresponds, as we have seen, that the related solutions go to infinity with *steady* oscillations.

It also of interest that the region inside which the functions (18) are considerably different from zero and inside which they display their oscillations is anyway of the *general order of magnitude* of the great axis of the assigned ellipse. The factor, multiplied with which the radius vector appears as argument of the constant-free function f, is — obviously — the reciprocal of a length, and this length is

$$\frac{K}{\sqrt{-2mE}} = \frac{K^2 l}{me^2} = \frac{h^2 l}{4\pi^2 m e^2} = \frac{a_l}{l} \ , \tag{21}$$

where a_l is the semi-major axis of the l-th elliptical trajectory. (The equations follow from (19) together with the known relation $E_l = -\frac{e^2}{2a_l^2}$).

The quantity (21) gives the order of magnitude of the region of the roots for small l and n; then it can be deduced that the roots of $f(x)$ are of order of magnitude one. This is naturally not the case if the coefficients of the polynomial are large numbers. I would like not to address now the more precise estimate of the roots, but I believe that through that the above claim will prove itself quite correct.

§ 3. It is natural to relate the function ψ to *a vibration process* occurring in the atom, which is more realistic than the today very often doubted electron trajectories. Originally, I also had the aim of motivating the new interpretation of the quantum prescription in this more expressive way, but then I have preferred the above neutral mathematical form because it allows to bring more clearly to light the essential. As the essential it seems to me that it does not come out anymore the secret "requirement of integerness", but this is, so to say, traced one step back: it has its basis in the finiteness and single-valuedness of a certain function of space.

I would also like now not yet to go closer this discussion of the possibilities interpretations of this vibration process, before some somehow more complicated cases are worked out with success in the new interpretation. It is not settled that the latter in its results will be a mere reproduction of the usual quantum theory. For example, the relativistic Kepler problem, if one goes through the calculation exactly according to the prescription given at the beginning, leads strangely to *half-integer* quantum *numbers* (radial and azimuthal quanta).

Nevertheless, let some more remarks about the idea of the vibration be allowed here. First of all, I would like not to leave unmentioned, that I owe the stimulation to these reflections in the first place to the brilliant thesis of Mr. Louis de Broglie[20]) and to the thought about the spatial distribution of those "phasewave", for which he has shown that always an *integer number* of them, measured along the trajectory, are allotted to each period or quasiperiod of the electron. The main difference is that de Broglie thinks of propagating waves, whereas we, if we put underneath our formulae the idea of vibration, are lead to standing proper oscillations. I have shown recently[21] that one can motivate Einstein's theory of gas by considering such standing proper oscillations, for which one applies the dispersion law of the de Broglie phasewave. The above considerations for the atom could have been represented as a generalisation of those thoughts on the model of gas.

If one interprets the individual functions (18), multiplied with a spherical harmonic of order n, as the description of processes of proper oscillations, then the quantity E must have something to do with the *frequency* of the process in question. Now, one is used to the fact that in oscillatory problems the "parameter" (usually called λ) is proportional to the *square* of the frequency. But, first, such an approach would lead in the present case, just for *negative* values of E, to *imaginary* frequencies, second their instinct tells to the quantum theorist, that the energy has to be proportional to the frequency itself and not to its square.

The contradiction resolves itself in the following way. For the "parameter" E of the variation equation (5) there is of course *no natural fundamental level* established, especially since the unknown function ψ appears multiplied, a part from E, also by a function of r which, upon a corresponding change in the fundamental level of E, can be altered by a constant. Consequently, the "expectation of the theoretician of the oscillations" must be corrected to the extent that not E itself — we called it so up to now and want to call it so also in the following — but E increased by a certain constant is expected to be proportional to the square of the frequency. Let now this constant be *very large* with respect to the modules of all occurring negative values of the energy [which are, of course, constrained by (15)]. Then, first, the frequencies become *real*, but, second, our values of the energy become, since they correspond only to relatively small *differences* in the frequencies, really very closely proportional to these differences in the frequency. On the other hand, this is all the "natural instinct" of the quantum theorist can demand, as long as the zero level of the *energy* is not established.

The interpretation that the frequency of the oscillatory process is given by something of the sort

$$\nu = C'\sqrt{C+E} = C'\sqrt{C} + \frac{C'}{2\sqrt{C}}E + \ldots \tag{22}$$

where C is a constant very large with respect to all the energies, has however another very estimable superiority. *It provides an understanding for Bohr's condition on the frequency.* According to the latter the *emission frequencies* are indeed proportional to the *energy differences*, then according to (22) also to the differences

[20]L. de Broglie, Ann. de Physique (10) 3. S. 22. 1925 (Thèses, Paris 1924)

[21]Published recently in the Physik. Zeitschr.

of the proper frequencies ν of that hypothetical vibration process. And indeed when the proper frequencies are all very large with respect to the emission frequencies, they match among themselves closely. The emission frequencies appear thus as deep "difference tones" of the proper oscillations themselves, which take place with much higher frequency. That by the shift of the energy from one into another normal oscillation *something* — I mean the lightwave — appears, to which as *frequency* is attributed the frequency *difference*, is very understandable; one needs only to imagine that the lightwave is causally linked with the *beat* that necessarily occur during the transition at every point of space and the frequency of the light is determined by the frequency with which the maximum of the intensity of the beat process recurs per second.

It might raise concerns that these conclusions are based on the relation (22) in its *approximated* form (upon development of the square root) by which Bohr's condition on the frequency itself acquires apparently the character of an approximated formula. However, that is only apparent and is completely avoided when one develops the *relativistic* theory by which in the first place a deeper understanding is conveyed. The large additive constant C is obviously most intimately connected with the rest energy mc^2 of the electron. Also the apparently *reiterated* and *independent* appearance of the constant h (which already was introduced in (20)) in the frequency condition is by the relativistic theory clarified, or better, avoided. But, unfortunately, its complete development encounters at the moment still certain, above touched, difficulties.

It is barely necessary to emphasise how much nicer would be the idea that in a quantum transition the energy shifts from one form of oscillation to another, than the idea of jumping electrons. The variation of the oscillatory form can constantly realise itself in space and time, according to the experience (canal rays attempts of W. Wien) it can well last as long as the emission process lasts: and nevertheless the proper frequencies will be determined, and together with them at once the oscillatory frequency, if during this transition the atom is subject to an electromagnetic field for a relatively short time, and only as long as the field is acting. This experimentally established fact causes, as it is well-known, up to now the biggest difficulties to the comprehension, cf. for example the discussion in the known attempt of solution of Bohr-Kramers-Slater.

Further one cannot certainly forget in the joy over the human proximity of all these things, that the idea that the atom vibrates, if it does not irradiate, specifically in the form of *one* proper oscillation, I say that this idea, *if* it must be retained, departs itself still very strongly from the *natural* picture of an oscillating system. Indeed, as it is well known, a macroscopic system does not certainly behave itself in such a way, but it delivers in general a potpourri of its proper frequencies. One cannot however establish rash their opinion about this point. Also a potpourri of proper frequencies for a single atom would make no difference, as long as at the same time no other beat frequencies appear as those to whose emission the atom is fit, according to the experience, *under conditions*. Also the simultaneous genuine emission of many of these spectral lines by the same atom does not contradict any experience. One could well think then that the atom oscillates only in normal conditions (and approximately in certain "metastable" conditions) with *one* frequency and right because of this it does *not* emit, because,

that is to say, no beats occur.

The *stimulation* would consist in a simultaneous state of excitation of one or more other proper frequencies through which then beats arise, that call for the light emission.

Under all circumstances I would like to believe that the eigenfunctions belonging to the *same* frequency are in general all excited simultaneously. Multiplicity of the eigenvalues corresponds namely, in the language of the present day theory, to the *degeneracy*. To the reduction of the quantisation of degenerate systems might correspond the arbitrary distribution of the energy over the eigenfunctions belonging to *one* eigenvalue.

Addition by the proofreading on 28. II. 1926.

For the case of the conservative systems of classical mechanics the variation task can be formulated in a way nicer than that at the beginning, without explicit relation to the Hamiltonian partial differential equation, as follows. Let $T(q,p)$ be the kinetic energy as function of the coordinates and of the momenta, V the potential energy, $d\tau$ the volume element of the configuration space "rationalised", that is not simply the product $dq_1, dq_2 \ldots dq_n$, but it divided by the square root of the discriminant of the quadratic form $T(q,p)$. (cf. Gibbs, Statistical Mechanics.) Then ψ should make the "Hamiltonian integral"

$$\int d\tau \left\{ K^2 T\left(q, \frac{\partial \psi}{\partial q}\right) + \psi^2 V \right\} \tag{23}$$

stationary under the *normalising additional condition*

$$\int \psi^2 d\tau = 1 \, . \tag{24}$$

The eigenvalues of this variation problem are, as it is well known, *the stationary values* of the integral (23) and deliver, according to our thesis, *the energy quantum levels*.

Regarding (14″) let it be still remarked that in the quantity α_2 one has essentially before themselves the known Sommerfeld's expression $-\frac{B}{\sqrt{A}} + \sqrt{C}$ (cf. "Atombau", 4. Aufl., S. 775).

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Translator's appendix

Applying Laplace transform to equation $(7'')$

Recall equation $(7'')$:

$$\frac{d^2U}{dr^2} + \left(\delta_0 + \frac{\delta_1}{r}\right)\frac{dU}{dr} + \left(\varepsilon_0 + \frac{\varepsilon_1}{r}\right)U = 0 \; . \tag{7''}$$

Let us represent $U(r)$ as follows:

$$U(r) = \int_L dz e^{zr} F(z) \; , \tag{25}$$

where L is a suitable integration path on the complex plane of the complex variable z. Plugging this representation into equation $(7'')$ one obtains:

$$\int_L dz e^{zr} [P(z) + Q(z)r] F(z) = 0 \; , \tag{26}$$

with

$$P(z) = \delta_1 z + \varepsilon_1 \; , \qquad Q(z) = z^2 + \delta_0 z + \varepsilon_0 \; . \tag{27}$$

Write:

$$r = \frac{d}{dz} e^{rz} \; , \tag{28}$$

and recast the transformed equation (26) as:

$$\int_L dz \frac{d}{dz} \left[e^{zr} Q(z) F(z)\right] - \int_L dz e^{zr} \left[-P(z)F(z) + Q(z)F'(z) + Q'(z)F(z)\right] = 0 \; , \tag{29}$$

where the prime denotes derivation with respect to z. The above equation is satisfied if:

$$-P(z)F(z) + Q(z)F'(z) + Q'(z)F(z) = 0 \; , \tag{30}$$

which is the first-order differential equation mentioned by Schrödinger in the text, and

$$\int_L dz \frac{d}{dz} \left[e^{zr} Q(z) F(z)\right] = 0 \; , \tag{31}$$

which will turn out to be condition (13). Let us write $Q(z)$ as:

$$Q(z) = z^2 + \delta_0 z + \varepsilon_0 = (z - c_1)(z - c_2) \; , \tag{32}$$

where c_1 and c_2 are, as explained in the text by Schrödinger, the roots of $Q(z)$ and therefore are related to δ_0 and ε_0 as follows:

$$c_1 + c_2 = -\delta_0 \; , \qquad c_1 c_2 = \varepsilon_0 \; . \tag{33}$$

Equation (30) can now indeed be solved by quadrature since it can be cast as follows:

$$\frac{F'}{F} = \frac{P - Q'}{Q} = \frac{z(\delta_1 - 2) + \varepsilon_1 + c_1 + c_2}{(z - c_1)(z - c_2)} \,, \tag{34}$$

and upon integration, a part an unimportant integration constant, we get:

$$F = (z - c_1)^{\alpha_1 - 1}(z - c_2)^{\alpha_2 - 1} \,, \tag{35}$$

with α_1 and α_2 given by equation (14′). With this F we reproduce equations (12) and (13).